

A Construction for Periodic ZCZ Sequences

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ABSTRACT. We introduce a construction for periodic zero correlation zone (ZCZ) sequences over roots of unity. The sequences share similarities to the perfect periodic sequence constructions of Liu, Frank, and Milewski. The sequences have two non-zero off-peak autocorrelation values which asymptotically approach $\pm 2\pi$, so the sequences are asymptotically perfect.

ZCZ sequences see applications in areas including broadband satellite IP networks [Zeng, 2005], CDMA systems [Suehiro, 1994][Fan, 1999b], quasi-synchronous code-division multiple-access (QS-CDMA) communication systems [De Gaudenzi, 1992], and watermarking [Van Schyndel, 2000][Tirkel, 2001].

Wolfmann [Wolfmann, 1992] introduced the idea of ZCZ binary sequences with the so-called *almost perfect sequences*, which have one non-zero off-peak autocorrelation value. Non-binary ZCZ sequences over roots of unity were first constructed by Suehiro [Suehiro, 1994]. Other types of ZCZ sequences have been constructed based on the existence of complementary pairs [Suehiro, 1994][Fan, 1999a][Matsufuji, 1999].

The periodic cross-correlation of the sequences, $\mathbf{a} = [a_0, a_1, \dots, a_{n-1}]$ and $\mathbf{b} = [b_0, b_1, \dots, b_{n-1}]$, for shift τ is given by

$$\theta_{\mathbf{a}, \mathbf{b}}(\tau) = \sum_{i=0}^{n-1} a_i b_{i+\tau}^*,$$

where $i + \tau$ is computed modulo n . The periodic autocorrelation of a sequence, \mathbf{s} for shift τ is given by $\theta_{\mathbf{s}}(\tau) = \theta_{\mathbf{s}, \mathbf{s}}(\tau)$. For $\tau \neq 0 \bmod n$, $\theta_{\mathbf{s}}(\tau)$ is called an *off-peak* autocorrelation. A sequence has good ZCZ autocorrelation if there are a small number of non-zero off-peak autocorrelation values, each of which are small in magnitude. Ideally, the non-zero values should be grouped closely together.

The periodic autocorrelation of a sequence, $\mathbf{s} = [s_0, s_1, \dots, s_{ld^2-1}]$, can be expressed in terms of the autocorrelation and cross-correlation of an array *associated* with \mathbf{s} [Heimiller, 1961][Frank, 1962][Mow, 1993]. The sequence \mathbf{s} has the *array orthogonality property* (AOP) for the *divisor* d if the array \mathbf{S} associated with \mathbf{s} has the following two properties:

1. For all τ , the periodic cross-correlation of any two distinct columns of \mathbf{S} is zero.
2. For $\tau \neq 0$, the sum of the periodic autocorrelation of all columns of \mathbf{S} is zero.

Any sequence with the AOP is perfect [Mow, 1993]. It is possible to relate the non-zero autocorrelation values of a sequence to the non-zero autocorrelation or cross-correlation values in the two conditions of the AOP. A non-zero in the autocorrelation for shift $\tau = q'd + r'$, ($r' < d$), corresponds to a non-zero in the cross correlation for shift $\kappa = q' + \left\lfloor \frac{r+r'}{d} \right\rfloor$, ($0 \leq r < d$) for the first condition of the AOP ($r' \neq 0$) or a non-zero in the sum of the autocorrelations for shift $\kappa = q' + \left\lfloor \frac{r}{d} \right\rfloor$, ($0 \leq r < d$) for the second condition of the AOP ($r' = 0$).

In most perfect sequence constructions, one proves the sequence has perfect autocorrelation by reducing the autocorrelation to a Gaussian summation. A Gaussian summation is given by $\sum_{k=0}^{n-1} \omega^{qk}$, where $\omega = e^{2\pi\sqrt{-1}/n}$ and $q \in \mathbb{Z}$. If $q \neq 0 \bmod n$, then the sum is zero.

We now introduce a construction for ZCZ sequences over roots of unity. This construction has similarities to the perfect periodic sequence construction of Liu [Liu, 2004] in that it uses the *floor* function within its index function.¹ Furthermore, it is similar to the perfect periodic sequence constructions of Frank

¹Note that our construction, like the construction of Liu, also holds for the *ceiling* function.

and Milewski [Frank, 1962][Milewski, 1983] in that it is formed by concatenating, row-by-row, a perfect array. Furthermore, this construction is very similar to the perfect sequence construction by the authors [Blake, 2012].

Let \mathbf{s} be a sequence of length $24(2n+1)$ over $6(2n+1)$ roots of unity, where $n \in \mathbb{N}$. Construct a $12(2n+1) \times 2$ array, \mathbf{S} , over $6(2n+1)$ roots of unity, where $\mathbf{S} = [S_{i,j}] = \omega^{\lfloor i(i+j)/2 \rfloor}$ and $\omega = e^{2\pi\sqrt{-1}/(6(2n+1))}$. The sequence, \mathbf{s} is constructed by enumerating, row-by-row, the array \mathbf{S} .

We show \mathbf{s} has periodic ZCZ autocorrelation. We begin by showing \mathbf{s} satisfies the first condition of the AOP. That is, we show that the cross-correlation of the two columns of \mathbf{S} is zero for every shift, κ . The cross-correlation of the two columns of \mathbf{S} is given by

$$\begin{aligned}
\theta_{S_{i,0}, S_{i,1}}(\kappa) &= \sum_{i=0}^{12(2n+1)-1} S_{i,0} S_{i+\kappa,1}^* \\
&= \sum_{i=0}^{12(2n+1)-1} \omega^{\lfloor \frac{i^2}{2} \rfloor} \omega^{-\lfloor \frac{(i+\kappa)^2 + i + \kappa}{2} \rfloor} \\
&= \sum_{i=0,2,4,\dots}^{12(2n+1)-1} \omega^{\lfloor \frac{i^2}{2} \rfloor} \omega^{-\lfloor \frac{(i+\kappa)^2 + i + \kappa}{2} \rfloor} + \sum_{i=1,3,5,\dots}^{12(2n+1)-1} \omega^{\lfloor \frac{i^2}{2} \rfloor} \omega^{-\lfloor \frac{(i+\kappa)^2 + i + \kappa}{2} \rfloor} \\
&= \sum_{k=0}^{6(2n+1)-1} \omega^{\lfloor \frac{(2k)^2}{2} \rfloor} \omega^{-\lfloor \frac{(2k+\kappa)^2 + 2k + \kappa}{2} \rfloor} + \sum_{k=0}^{6(2n+1)-1} \omega^{\lfloor \frac{(2k+1)^2}{2} \rfloor} \omega^{-\lfloor \frac{(2k+1+\kappa)^2 + 2k+1 + \kappa}{2} \rfloor} \\
&= \omega^{\lfloor \frac{\kappa^2 + \kappa}{2} \rfloor} (1 + \omega^{-\kappa-1}) \sum_{k=0}^{6(2n+1)-1} \omega^{-(2\kappa+1)k}.
\end{aligned}$$

The (Gaussian) summation above is zero as $-2\kappa-1 \neq 0 \pmod{6(2n+1)}$. Thus, \mathbf{s} satisfies the first condition of the AOP, which implies the autocorrelation of the sequence, \mathbf{s} is zero for $\tau \neq 0 \pmod{2}$.

We now show \mathbf{s} satisfies the second condition of the AOP with the exception of two off-peak shifts. That is, we show that the sum of the autocorrelation of the two columns of \mathbf{S} is zero for all but two shifts.

$$\theta_{S_{i,0}}(\kappa) + \theta_{S_{i,1}}(\kappa) = \sum_{i=0}^{12(2n+1)-1} S_{i,0} S_{i+\kappa,0}^* + \sum_{i=0}^{12(2n+1)-1} S_{i,1} S_{i+\kappa,1}^* \quad (1) + (2)$$

From which (1) becomes

$$\begin{aligned}
&\sum_{i=0}^{12(2n+1)-1} \omega^{\lfloor \frac{i^2}{2} \rfloor} \omega^{-\lfloor \frac{(i+\kappa)^2}{2} \rfloor} \\
&= \sum_{i=0,2,4,\dots}^{12(2n+1)-1} \omega^{\lfloor \frac{i^2}{2} \rfloor} \omega^{-\lfloor \frac{(i+\kappa)^2}{2} \rfloor} + \sum_{i=1,3,5,\dots}^{12(2n+1)-1} \omega^{\lfloor \frac{i^2}{2} \rfloor} \omega^{-\lfloor \frac{(i+\kappa)^2}{2} \rfloor} \\
&= \sum_{k=0}^{6(2n+1)-1} \omega^{\lfloor \frac{(2k)^2}{2} \rfloor} \omega^{-\lfloor \frac{(2k+\kappa)^2}{2} \rfloor} + \sum_{k=0}^{6(2n+1)-1} \omega^{\lfloor \frac{(2k+1)^2}{2} \rfloor} \omega^{-\lfloor \frac{(2k+1+\kappa)^2}{2} \rfloor} \\
&= \left(\omega^{-\lfloor \frac{\kappa^2}{2} \rfloor} + \omega^{-\lfloor \frac{(1+\kappa)^2}{2} \rfloor} \right) \sum_{k=0}^{6(2n+1)-1} \omega^{-2\kappa k}.
\end{aligned}$$

Similarly, (2) becomes

$$\begin{aligned}
& \sum_{i=0}^{12(2n+1)-1} \omega^{\lfloor \frac{i^2+i}{2} \rfloor} \omega^{-\lfloor \frac{(i+\kappa)^2+i+\kappa}{2} \rfloor} \\
&= \sum_{i=0,2,4,\dots}^{12(2n+1)-1} \omega^{\lfloor \frac{i^2+i}{2} \rfloor} \omega^{-\lfloor \frac{(i+\kappa)^2+i+\kappa}{2} \rfloor} + \sum_{i=1,3,5,\dots}^{12(2n+1)-1} \omega^{\lfloor \frac{i^2+i}{2} \rfloor} \omega^{-\lfloor \frac{(i+\kappa)^2+i+\kappa}{2} \rfloor} \\
&= \sum_{k=0}^{6(2n+1)-1} \omega^{\lfloor \frac{(2k)^2+2k}{2} \rfloor} \omega^{-\lfloor \frac{(2k+\kappa)^2+2k+\kappa}{2} \rfloor} + \sum_{k=0}^{6(2n+1)-1} \omega^{\lfloor \frac{(2k+1)^2+2k+1}{2} \rfloor} \omega^{-\lfloor \frac{(2k+1+\kappa)^2+2k+1+\kappa}{2} \rfloor} \\
&= \left(\omega^{-\lfloor \frac{\kappa^2+\kappa}{2} \rfloor} + \omega^{-\lfloor \frac{\kappa^2+3\kappa}{2} \rfloor} \right) \sum_{k=0}^{6(2n+1)-1} \omega^{-2\kappa k}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\theta_{S_{i,0}}(\kappa) + \theta_{S_{i,1}}(\kappa) &= (1) + (2) \\
&= \left(\omega^{-\lfloor \frac{\kappa^2}{2} \rfloor} + \omega^{-\lfloor \frac{(1+\kappa)^2}{2} \rfloor} + \omega^{-\lfloor \frac{\kappa^2+\kappa}{2} \rfloor} + \omega^{-\lfloor \frac{\kappa^2+3\kappa}{2} \rfloor} \right) \sum_{k=0}^{6(2n+1)-1} \omega^{-2\kappa k}.
\end{aligned}$$

This (Gaussian) sum is non-zero when $\kappa = 3(2n+1)$, or $\kappa = 6(2n+1)$, or $\kappa = 9(2n+1)$. When $\kappa = 6(2n+1)$, $\omega^{-\lfloor \frac{\kappa^2}{2} \rfloor} + \omega^{-\lfloor \frac{(1+\kappa)^2}{2} \rfloor} + \omega^{-\lfloor \frac{\kappa^2+\kappa}{2} \rfloor} + \omega^{-\lfloor \frac{\kappa^2+3\kappa}{2} \rfloor} = 0$. So \mathbf{s} satisfies the second condition of the AOP at all non-zero shifts except $\kappa = 3(2n+1)$ and $\kappa = 9(2n+1)$.

Thus, the autocorrelation of \mathbf{s} has two non-zero values at shifts $\tau = 6(2n+1)$ and $\tau = 18(2n+1)$. The two non-zero autocorrelation values are equal, and are given by

$$(-1)^{n+1} 12(2n+1) \sin\left(\frac{\pi}{6(2n+1)}\right),$$

which asymptotically approaches 2π for odd n and -2π for even n . As a result, for arbitrarily large lengths, the ratio of the non-zero off-peak autocorrelation values and the length approaches zero. So, in a sense, \mathbf{s} is an asymptotically perfect sequence.

References

- [Blake, 2012] S. T. Blake, A. Z. Tirkel, “A Construction for Perfect Periodic Autocorrelation Sequences”, submitted for publication, December 2012
- [De Gaudenzi, 1992] R. De Gaudenzi, C. Elia, and R. Viola, “Bandlimited quasisynchronous CDMA: A novel satellite access technique for mobile and personal communication systems”, *IEEE J. Sel. Areas Commun.*, vol. 10, pp. 328-343, February 1992
- [Fan, 1999a] P. Z. Fan, N. Suehiro, N. Kuroyanagi, X. M. Deng, “A class of binary sequences with zero correlation zone”, *IEE Electron. Lett.*, vol. 35, pp. 777-779, May 1999
- [Fan, 1999b] P. Z. Fan, N. Suehiro, N. Kuroyanagi, “A novel interference-free CDMA system”, *Proc. Int. Symp. Personal, Indoor, Mobile Radio Communications*, Osaka, Japan, pp. 440-444, September 1999

- [Frank, 1962] R. L. Frank, S. A. Zadoff and R. Heimiller, "Phase Shift Pulse Codes with Good Periodic Correlation Properties", *IRE Transactions on Information Theory*, vol. 8, no. 6, pp. 381-382, October 1961
- [Heimiller, 1961] R. C. Heimiller, "Phase Shift Pulse Codes with Good Periodic Correlation Properties", *IRE Transactions on Information Theory*, vol. 7, no. 4, pp. 254-257, October 1961
- [Liu, 2004] Y. Liu and P. Fan, "Modified Chu sequences with smaller alphabet size", *Electronics Letters*, vol. 40, no. 10, May 2004
- [Matsufuji, 1999] S. Matsufuji, N. Suehiro, N. Kuroyanagi, P. Z. Fan, K. Takatsukasa, "A binary sequences pair with zero correlation zone derived from complementary pairs", *Proc. Int. Symp. Communication Theory and Application (ISCTA99)*, Ambleside, Lancashire, U.K., pp. 223-224, July 1999
- [Milewski, 1983] A. Milewski, "Periodic Sequences with Optimal Properties for Channel Estimation and Fast Start-Up Equalization", *IBM Journal of Research and Development*, vol. 27, no. 5, pp. 426-431, September 1983
- [Mow, 1993] W. H. Mow, "A Study of Correlation of Sequences", PhD, Department of Information Engineering, The Chinese University of Hong Kong, 1993
- [Suehiro, 1994] N. Suehiro, "A signal design without co-channel interference for approximately synchronized CDMA systems", *IEEE J. Select. Areas Commun.*, vol. 12, pp. 837-841, June 1994
- [Tirkel, 2001] A. Z. Tirkel, T. E. Hall, "A Unique Watermark for Every Image", *IEEE Multimedia*, vol. 8, no. 4, pp. 30-37, October 2001
- [Van Schyndel, 2000] R. G. Van Schyndel, A. Z. Tirkel, I. D. Svalbe, T. E. Hall, C. F. Osborne, "Spread-Spectrum Digital Watermarking Concepts and Higher Dimensional Array Constructions", *First International Online Symposium on Electronics Engineering*, July 2000
- [Wolfmann, 1992] J. Wolfmann, "Almost perfect autocorrelation sequences", *IEEE Trans. Inform. Theory*, vol. 38, no. 4, pp. 1412-1418, 1992
- [Zeng, 2005] X. Zeng, L. Hu, Q. Liu, "New Sequence Sets with Zero-Correlation Zone", <http://arxiv.org/pdf/cs/0508115.pdf>, 2005